AIMS | African Institute for Mathematical Sciences SOUTH AFRICA

2022

Risk Management and Insurance

Tutorial 1. Arithmetic and Geometric Series

Question 1

Which series are arithmetic and which are geometric? What is the common difference or common ratio?

a) $3 + 6 + 12 + 24 + \ldots$

b) $5 + 8 + 11 + 14 + \ldots$

c) $100 + 98 + 96 + 94 + \ldots$

d) $10 + 5 + 2.5 + 1.25 + \ldots$

Question 2

Prove that the sum of the first n terms of an arithmetic series is:

$$S_n = (a_1 + a_n)\frac{n}{2}$$

Question 3

Consider the geometric series

$100 + 80 + 64 + \ldots..$

Find the 20th term, the sum of the first 20 terms, and the sum of the infinite series.

Question 4

Prove that the sum of the first n terms of a geometric series is:

$$S_n = a_1 \frac{1-q^n}{1-q}$$

Question 5

Show that the sum of an infinite geometric series is:

$$S_\infty = \frac{a_1}{1-q} \qquad |q| < 1$$

Question 6

Gauss's primary school teacher asked the class to add up all numbers from 1 to 100, probably hoping that this would keep it busy for some time. To the teacher's astonishment, young Gauss almost immediately came up with the correct answer. How did he do it?

Question 7

Change the following periodic decimals into fractions.

a) 0.1212… b) 1.1333…

Hint: Periodic decimals can be written as infinite geometric series.

Question 8

Show that $0.9999\ldots = 1$

Question 9

An Indian king was so impressed by chess that he offered its inventor an award. The mathematician responded that the king should put one grain of rice on the first square, two on the second, four on the third and so on for all 64 squares, always doubling the number of grains from one square to the next. How many grains of rice would the king have to pay?

Question 10

Show that the series $\sum_{i=1}^{\infty} \frac{i}{2^i}$ converges to 2.

Question 11

What process describes the behavior of real GDP better, an arithmetic or geometric one? What about the price of a share?

Tutorial 1

Question 1

a) geometric, $q = 2$

b) arithmetic, $d = 3$

c) arithmetic, $d = -2$

d) geometric, $q = 0.5$

Question 2

The sum of the first n terms can be written in two ways:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + (a_n - 2d) + (a_n - d) + a_n$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + \cdots + (a_1 + 2d) + (a_1 + d) + a_1$$

Add the two expressions term by term:

$$2S_n = (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n)$$
$$2S_n = n(a_1 + a_n)$$
$$S_n = (a_1 + a_n)\frac{n}{2}$$

(A more formal proof uses mathematical induction.)

Question 3

Geometric series:

$$a_1 + a_1 q + a_1 q^2 + a_1 q^3 + \cdots$$

Here: $a_1 = 100$, $q = 0.8$

20th term:

$$a_n = a_1 q^{n-1}$$
$$= 100 \cdot 0.8^{(20-1)} = \underline{\underline{1.44}}$$

Sum of the first 20 terms:

$$S_n = a_1 \frac{1-q^n}{1-q}$$
$$= 100 \, \frac{1-0.8^{20}}{1-0.8} = \underline{\underline{494.23}}$$

Sum of the infinite geometric series:

$$S_\infty = \frac{a_1}{1-q}$$
$$= \frac{100}{1-0.8} = 500$$

Note that the sum of the first 20 terms is close to the sum of the infinite series.

Question 4

Sum of the 1st n terms

$$S_n = a_1 + a_1 q + a_1 q^2 + a_1 q^3 + \cdots\cdots + a_1 q^{n-2} + a_1 q^{n-1}$$
$$q S_n = a_1 q + a_1 q^2 + a_1 q^3 + a_1 q^4 + \cdots\cdots + a_1 q^{n-1} + a_1 q^n$$

Subtract the two expressions term by term:

$$S_n - q S_n = a_1 - a_1 q^n$$
$$S_n (1-q) = a_1 (1-q^n)$$
$$S_n = a_1 \frac{1-q^n}{1-q}$$

(or use mathematical induction)

Question 5

Suppose $|q| < 1$. Then,

$$S_\infty = \lim_{n \to \infty} \left[a_1 \frac{1-q^n}{1-q} \right] = \frac{a_1}{1-q}$$

Question 6

$$1 + 2 + 3 + \cdots + 100$$

This is an arithmetic series with $a_1 = 1$, $a_n = 100$, $n = 100$

$$S_{100} = (a_1 + a_n)\frac{n}{2} = (1 + 100)\frac{100}{2} = \underline{\underline{5050}}$$

Gauss probably did not know the sum formula for arithmetic series when he was in primary school but he may have noticed the following pattern:

$$\left.\begin{array}{rcl} 1 + 100 & = & 101 \\ 2 + 99 & = & 101 \\ 3 + 98 & = & 101 \\ & \vdots & \\ 50 + 51 & = & 101 \end{array}\right\} \quad 50 \times 101 = \underline{\underline{5050}}$$

Tutors:

Give full mark for this question if a student <u>either</u> uses the sum formula for an arithmetic series <u>or</u> a pattern like Gauss.

Question 7

a) $0,121212\ldots = 0.12 + 0.0012 + 0.000012 + \ldots$

This is a geometric series with $a_1 = 0.12$ and $q = 0.01$

$$S_\infty = \frac{a_1}{1-q} = \frac{0.12}{1-0.01} = \frac{0.12}{0.99} = \frac{12}{99} = \underline{\underline{\frac{4}{33}}}$$

b) $1.1333\ldots = 1 + \frac{1}{10} + (0.03 + 0.003 + 0.0003 + \ldots)$

$$= 1 + \frac{1}{10} + \frac{0.03}{1-0.1} = 1 + \frac{1}{10} + \frac{0.03}{0.9}$$

$$= 1 + \frac{1}{10} + \frac{3}{90} = 1 + \frac{9}{90} + \frac{3}{90} = 1 + \frac{12}{90}$$

$$= \underline{\underline{1\frac{2}{15}}}$$

Every periodic decimal can be written as a <u>ratio</u>. ($\longrightarrow$ <u>rational</u> numbers)

Question 8

$$0.999\ldots = 0.9 + 0.09 + 0.009 + \ldots$$

$$= \frac{0.9}{1-0.1} = 1$$

Question 9

This is a geometric series with $a_1 = 1$, $q = 2$, $n = 64$.

$$S_{64} = a_1 \frac{1-q^n}{1-q} = 1 \, \frac{1-2^{64}}{1-2}$$

$$= 18{,}446{,}744{,}073{,}709{,}551{,}615$$

This is an enormous figure that amounts to about 400 times the current world production of rice (assuming 1 grain of rice = $1/64$ g)

EXCEL is unable to handle so many digits and gives:

$$1.844674 \times 10^{19}$$

$$18{,}446{,}744{,}073{,}709{,}\underline{600{,}000}$$

Question 10

1

geometric series

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1/2}{1-1/2} = 1$$

$$\frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1/2^2}{1-1/2} = \frac{1}{2}$$

$$\frac{1}{2^3} + \cdots = \frac{1/2^3}{1-1/2} = \frac{1}{2^2}$$

2

$$\sum_{i=1}^{n} \frac{i}{2^i} = \frac{1}{1-\frac{1}{2}} = 2$$

geometric series

Question 11

Arithmetic processes are additive and geometric ones are multiplicative. It is commonly thought that both real GDP and share prices follow geometric processes.

- Economists consider the (relative) growth rate of real GDP, which is measured in percent.

$$GDP_{2015} = (1+g)GDP_{2014}$$

$$g = \frac{GDP_{2015} - GDP_{2014}}{GDP_{2014}}$$

The annual growth rate g is measured in %. For example: 0.03 →3%.

- Investors are mostly interested in the capital gain on a share, measured in percent.

$$S_t = (1+g)S_{t-1}$$

$$g = \frac{S_t - S_{t-1}}{S_{t-1}}$$

The increase in the share price S from one period to the next is measured in %.