AIMS | African Institute for Mathematical Sciences SOUTH AFRICA

2022

Risk Management and Insurance

Tutorial 4. Annuity and Perpetuity, Present Value

All discrete payments are made at the end of the payment period (ordinary annuity and perpetuity).

Question 1

Find the sum of the first n terms and of the infinite geometric series

$$\frac{1}{(1+R)}+\frac{1}{(1+R)^2}+\frac{1}{(1+R)^3}+\cdots$$

Question 2

Show that the present value of an annuity that runs for T years is:

$$V_0 = V\frac{1-(1+R)^{-T}}{R}$$

Hint: Use the result of Question 1.

Question 3

Show that the present value of V dollars per year paid continuously for T years is:

$$V(0)=\frac{V}{R}\left(1-e^{-RT}\right)$$

Question 4

Show that the present value of a perpetuity of V dollars per year is:

$$V_0 = \frac{V}{R}$$

Does it matter whether the perpetuity is paid at the end of the year or continuously?

Question 5

Equation of value: A person owes \$ 10,000 due in one year and \$ 30,000 due in 4 years. He or she agrees to pay \$ 20,000 today and the remainder in 2 years. What is the payment in 2 years if the interest rate is 5%, compounded semiannually?

Hint: Set up the equation of value with a focal date of 2 years. Does the focal date matter?

Question 6

Sinking fund: A worker plans to retire in 30 years. How much must he save every year so that he has \$ 600,000 at retirement? Assume the net return on savings is 4% per year, annually compounded.

Question 7

Mortgage: The following table shows the monthly repayments for a 25 year mortgage. The loan values range from \$ 100,000 to \$ 500,000 and three interest rates are given.

Loan	6.32% (current)	6.57% (imminent)	7.82% (forecast)
\$100,000	\$664	\$680	\$760
\$150,000	\$996	\$1019	\$1140
\$200,000	\$1328	\$1359	\$1520
\$250,000	\$1660	\$1699	\$1900
\$300,000	\$1992	\$2039	\$2280
\$350,000	\$2324	\$2379	\$2660
\$400,000	\$2656	\$2718	\$3040
\$450,000	\$2988	\$3058	\$3420
\$500,000	\$3320	\$3398	\$3800

How is this table calculated? Recalculate the monthly payment for a loan of \$ 200,000 with an interest rate of 6.32%. Use monthly interest compounding.

Present Value Calculation:

Question 8

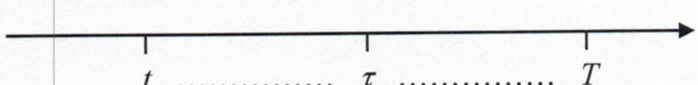
Evaluate the following expression at $z = 5$:

$$\frac{d}{dz}\int_{2}^{z}(x^2+3)dx$$

What does this expression show?

Question 9

Notation: The time index τ runs from t to T.



a) Suppose an investment produces a continuous stream of income over 10 years at a rate of \$20,000 per year and the interest rate is 6% per year, continuously compounded. Determine the present value.

b) Determine the rate of increase in the value of the investment project if its life is extended by an instant, dP/dT.

c) What is the present value in question (a) if the stream of income lasts forever?

Question 10

Consider an oil field that produces a diminishing cash flow that falls at the rate s as the field is exploited.

$$C(t) = e^{-st}C(0)$$

With a time horizon of T years, the field's present value is:

$$P(0) = \int_0^T e^{-rt}C(t)dt = \int_0^T e^{-(r+s)t}C(0)dt$$

r is the interest rate.

a) Calculate the field's present value for $C(0)$ = \$ 100,000,000, r = 0.12, s = 0.05, and T = 20 years.

b) Calculate the rate of change in the field's present value, dP/dT, if it is operated for an instant beyond time T.

Tutorial 4

Question 1

This is a geometric series with:

$$a_1 = \frac{1}{1+R}, \quad q = \frac{1}{1+R}$$

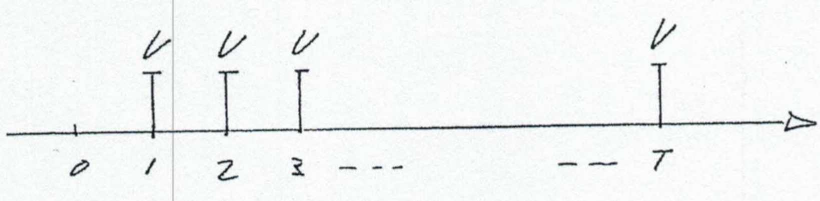
$$S_n = a_1 \frac{1-q^n}{1-q} = \frac{1}{1+R} \frac{1-\left(\frac{1}{1+R}\right)^n}{1-\frac{1}{1+R}}$$

$$= \frac{1-\left(\frac{1}{1+R}\right)^n}{(1+R)-1} = \underline{\underline{\frac{1-(1+R)^{-n}}{R}}}$$

$$S_\infty = \lim_{n\to\infty} \frac{1-(1+R)^{-n}}{R} = \underline{\underline{\frac{1}{R}}}$$

(Substituting into $S_\infty = \frac{a}{1-q}$ yields the same result.)

Question 2



Present value of annuity:

$$V_0 = \frac{V}{1+R} + \frac{V}{(1+R)^2} + \frac{V}{(1+R)^3} + \cdots + \frac{V}{(1+R)^T}$$

$$= V\left[\frac{1}{1+R} + \frac{1}{(1+R)^2} + \frac{1}{(1+R)^3} + \cdots + \frac{1}{(1+R)^T}\right]$$

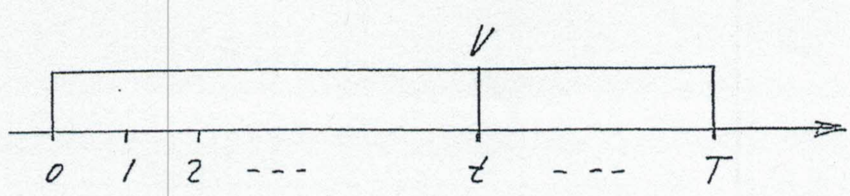
The bracket includes the 1st T terms of a geometric series.

Using the result from Question 1:

$$V_0 = \frac{V}{R}\left(1 - (1+R)^{-T}\right)$$

Question 3

Annuity (continuous time):



Present value:

$$V_0 = \int_0^T V e^{-Rt} dt = V \int_0^T e^{-Rt} dt$$

$$= V\left[\frac{-1}{R} e^{-Rt}\right]_0^T = V\left[\frac{-1}{R} e^{-RT} - \frac{(-1)}{R} e^{-R0}\right]$$

$$= V\left[\frac{-1}{R} e^{-RT} - \frac{(-1)}{R}\right] = \underline{\underline{\frac{V}{R}\left(1 - e^{-RT}\right)}}$$

<u>Question 4</u>

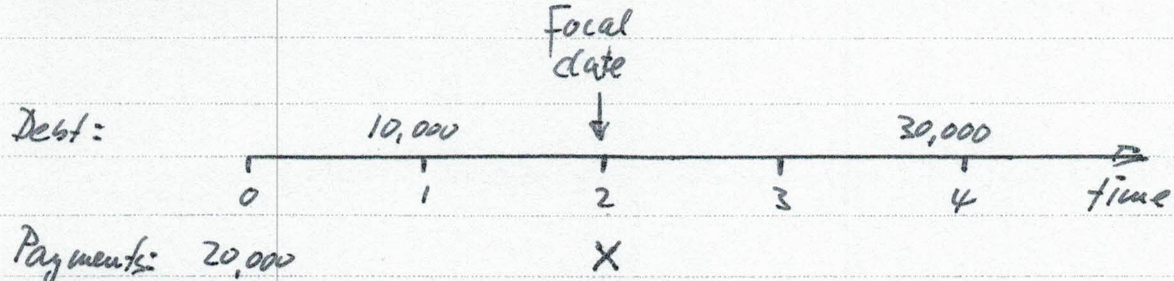
Take the limit of the annuity.

$$V_0 = \lim_{T \to \infty} \frac{V}{R}\left(1 - (1+R)^{-T}\right) = \frac{V}{R} \qquad (i)$$

$$V_0 = \lim_{T \to \infty} \frac{V}{R}\left(1 - e^{-RT}\right) = \frac{V}{R} \qquad (ii)$$

It does not matter whether the perpetuity is paid annually (i) or continuously (ii).

Question 5



Equation of value:

$$\underbrace{20{,}000(1.025)^{4} + X}_{\text{value of payments at focal date}} = \underbrace{10{,}000(1.025)^{2} + 30{,}000(1.025)^{-4}}_{\text{value of debts at focal date}}$$

Solve for X:

$$X = 10{,}000(1.025)^{2} + 30{,}000(1.025)^{-4} - 20{,}000(1.025)^{4}$$

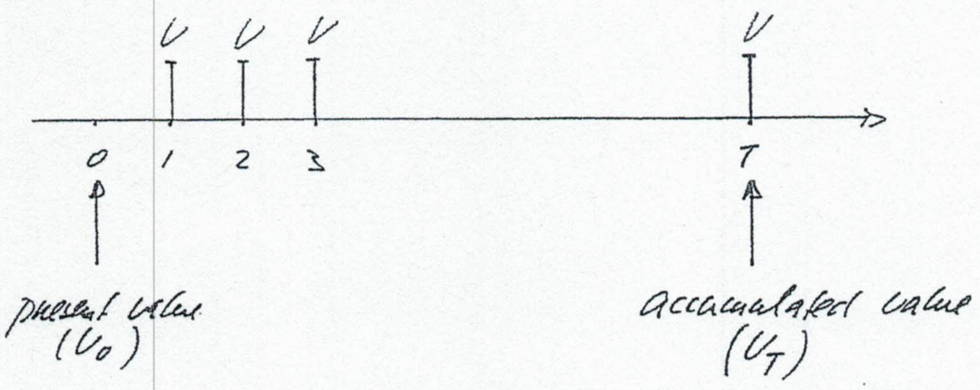
$$= \$\,15{,}608.51$$

Discuss the following point with the students:

It seems natural to use year 2 as focal date for the equation of value in this questions but with compound interest any date could be used.

However, with simple interest the focal date in the equation of value matters for the solution!

Question 6



V_T can be calculated by applying interest compounding to V_0.

$$V_T = V_0 (1+R)^T = V \frac{(1-(1+R)^{-T})}{R} (1+R)^T$$

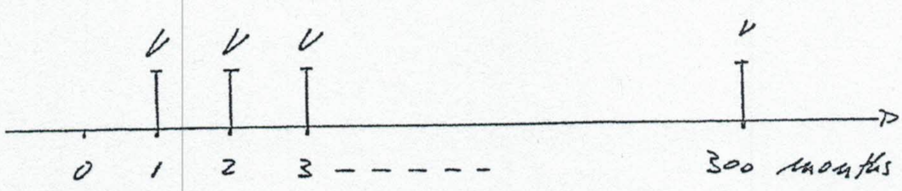
$$V_T = V \frac{((1+R)^T - 1)}{R} \qquad \text{(accumulated value)}$$

Solve for V:

$$V = \frac{V_T \cdot R}{(1+R)^T - 1}$$

$$= \frac{600'000 \times 0.04}{(1+0.04)^{30} - 1} = \$\ 10'698.06$$

Question 7



The present value of all payments V must be equal to the principal of the mortgage.

Following US practice, the table uses monthly interest compounding.

It is assumed that the interest rate will be constant.

$$\text{Principal} = \frac{V}{1+\frac{R}{12}} + \frac{V}{\left(1+\frac{R}{12}\right)^2} + \frac{V}{\left(1+\frac{R}{12}\right)^3} + \text{----} \frac{V}{\left(1+\frac{R}{12}\right)^{300}}$$

$$= V\left[\frac{1}{1+\frac{R}{12}} + \frac{1}{\left(1+\frac{R}{12}\right)^2} + \text{---} \frac{1}{\left(1+\frac{R}{12}\right)^{300}}\right]$$

Use sum formula for the 1st n terms of a geometric series:

$$\text{Principal} = V \frac{1}{1+\frac{R}{12}} \frac{1-\left(\frac{1}{1+\frac{R}{12}}\right)^{300}}{1-\frac{1}{1+\frac{R}{12}}} = V \frac{1-\frac{1}{\left(1+\frac{R}{12}\right)^{300}}}{\left(1+\frac{R}{12}\right)-1}$$

$$\text{Principal} = V \frac{1-\frac{1}{\left(1+\frac{R}{12}\right)^{300}}}{\frac{R}{12}}$$

$$V = \frac{R}{12} \frac{\text{Principal}}{1-\frac{1}{\left(1+\frac{R}{12}\right)^{300}}}$$

<u>Example:</u> Principal = 200'000 , $R = 6.32\%$

$$V = \frac{0.0632}{12} \frac{200'000}{1-\frac{1}{\left(1+\frac{0.0632}{12}\right)^{300}}} = \underline{\underline{\$ 1328.006}}$$

Question 8

The fundamental theorem of calculus is:

$$\frac{d}{dz}\int_a^z f(x)\,dx = f(z)$$

Here, this is:

$$\frac{d}{dz}\int_2^z \left(x^2+3\right)dx = z^2+3 = 5^2+3 = 28$$

The expression shows the rate of change of the value of the integral if the upper limit (5) is increased.

Question 9

a)

$$P(t) = \int_0^{10} 20'000\, e^{-0.06\tau}\, d\tau$$

$$= 20'000 \int_0^{10} e^{-0.06\tau}\, d\tau = 20'000 \left[-\frac{e^{-0.06\tau}}{0.06} \right]_0^{10}$$

$$= 20'000 \left[-\frac{e^{-0.06\times 10}}{0.06} + \frac{e^{-0.06\times 0}}{0.06} \right]$$

$$= \frac{20'000}{0.06} \left[1 - e^{-0.06\times 10} \right] = \$\ 150'396$$

The present value of an annuity that pays a continuous income stream V is

$$P(t) = \frac{V}{R} \left[1 - e^{-RT} \right]$$

4-10

b)

$$\frac{dP}{dT} = \frac{d}{dT}\int_t^T 20'000\, e^{-0.06(\tau - t)}\, d\tau$$

Fundamental theorem of calculus (version 1)

$$= 20'000\, e^{-0.06(T-t)} = 20'000\, e^{-0.06(10-0)}$$

$$= \$\ 10976$$

The rate of change of the present value equals the present value of the amount earned at time T.

c) Improper integral:

$$P(t) = \lim_{b \to \infty} \int_0^b 20'000\, e^{-0.06\tau}\, d\tau$$

See question (a)

$$= \lim_{b \to \infty} \frac{20'000}{0.06}\left[1 - e^{-0.06b}\right]$$

$$= \underbrace{\frac{20'000}{0.06}}_{\frac{V}{R}} = \$\ 333'333$$

Question 10

Consider an oil field that produces a diminishing cash flow that falls at rate s as the field is exploited.

$$C(t) = e^{-st}C(0)$$

With a time horizon of T years, the field's present value is:

$$P(0) = \int_0^T e^{-(r+s)t}C(0)dt$$

r is the interest rate.

a) Calculate the field's present value for $C(0)$ = \$ 100,000,000, $r = 0.12$, $s = 0.05$, and $T = 20$ years.

4C

$$P(0) = \int_0^{20} e^{-(0.12+0.05)t}\ 100\ \text{mio}\ dt$$

$$= 100\ \text{mio} \int_0^{20} e^{-0.17t}\ dt = 100\ \text{mio}\left[-\frac{e^{-0.17t}}{0.17}\right]_0^{20}$$

$$= 100\ \text{mio}\left[-\frac{e^{-0.17\times 20}}{0.17} - \left(-\frac{e^{0}}{0.17}\right)\right] = \$\ 568,603,958.85$$

b) Calculate the rate of change in the present value if the field is operated for an instant beyond tir-- T

5C

Using the first version of the fundamental theorem of calculus:

$$\frac{dP}{dT} = e^{-(r+s)T}C(0) = e^{-(0.12+0.05)20} \cdot 100\ \text{mio}$$

$$= \$\ 3,337,327.00$$