AIMS | African Institute for Mathematical Sciences SOUTH AFRICA

2022

Risk Management and Insurance

Tutorial 8. Risk Aversion and Insurance Demand

Question 1

Consider the CRRA utility function.

$$v(W) = \frac{W^{1-\gamma} - 1}{1-\gamma}$$

a) Compute the elasticity of marginal utility with regard to a change in wealth.

$$\varepsilon_{MU,W} = \frac{dv'}{dW}\frac{W}{v'} = \frac{v'' W}{v'}$$

b) What is the coefficient of relative risk aversion?

c) Is constant relative risk aversion a desirable property of a utility function?

Question 2

Show that the CRRA utility function simplifies to lnW_t if γ approaches 1.

Hint: Use L'Hôpital's rule.

Question 3

Does constant relative risk aversion imply decreasing absolute risk aversion? Does decreasing absolute risk aversion imply constant relative risk aversion?

Question 4

Consider an individual who faces a risky prospect:

Utility function: $u(W) = W^{0.5}$

Initial wealth: 10

Risky prospect: (-6, 0.5; +6, 0.5)

a) Determine the individual's exact certainty equivalent and risk premium.

b) Use the Arrow-Pratt formula to obtain an approximation of the risk premium.

c) What is the individual's maximum willingness to pay for insurance?

Tutorial 8

Question 1

a) With CRRA utility:

$$U' = W^{-\rho}$$

$$U'' = -\rho W^{-(1+\rho)}$$

Elasticity of marginal utility with respect to a change in wealth

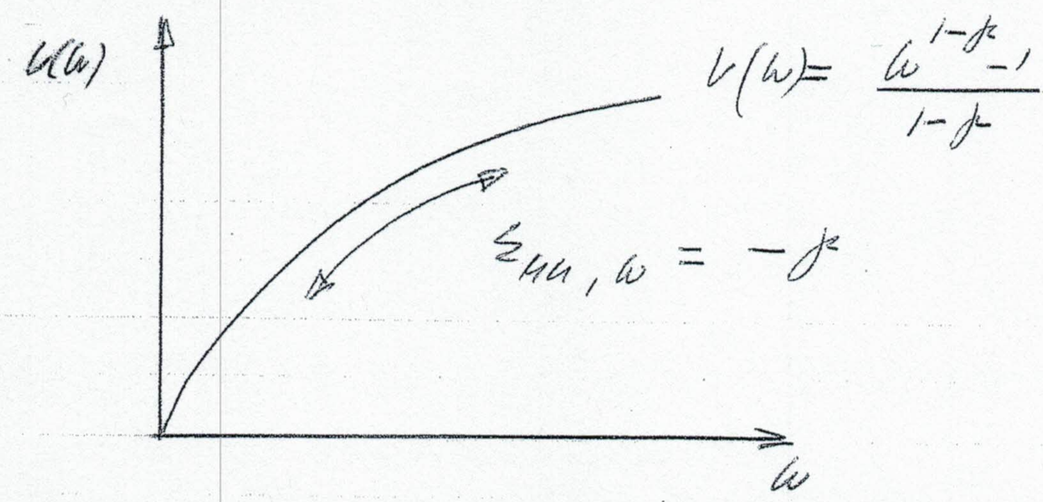
$$\varepsilon_{MU,W} = \frac{dU'}{dW}\frac{W}{U'} = \frac{U''\,W}{U'}$$

$$= \frac{-\rho W^{-(1+\rho)}\,W}{W^{-\rho}} = -\rho \quad < 0$$

The wealth elasticity of MU ($\varepsilon_{MU,W}$) measures the sensitivity of MU with respect to a change in wealth.

An increase in wealth reduces MU.

With CRRA utility, $\varepsilon_{MU,W}$ is independent of wealth.



b) The Arrow-Pratt coefficient of relative risk aversion is the negative of $\varepsilon_{u'u',x}$.

$$R_R = -\frac{v''(w)\, w}{v'(w)} = \rho$$

c) It is an empirical question whether relative risk aversion is independent of wealth.

CRRA (and CARA) are convenient for analytical work.

Question 2

Limits of ratios of functions sometimes misbehave.

$$\lim_{x \to a} \frac{m(x)}{n(x)} \longrightarrow \frac{0}{0}, \frac{\infty}{\infty}$$

L'Hôpital's rule says:

$$\lim_{x \to a} \frac{m(x)}{n(x} = \lim_{n \to a} \frac{m'(x)}{n'(x)}$$

If this does not work, use 2nd, 3rd, --- order derivatives.

CRRA utility function:

$$\lim_{\rho \to 1} \frac{w^{1-\rho} - 1}{1-\rho} \longrightarrow \frac{0}{0}$$

Base conversion:

$$w^{1-\rho} = \left(e^{\ln w}\right)^{1-\rho} = e^{(1-\rho)\ln w}$$

Therefore:

$$V(w) = \frac{w^{1-\rho} - 1}{1-\rho} = \frac{e^{(1-\rho)\ln w} - 1}{1-\rho}$$

Apply L'Hôpital's rule (use 1st order derivatives of numerator and denominator):

$$\lim_{\rho \to 1} v(w) = \lim_{\rho \to 1} \frac{e^{(1-\rho)\ln w}(-\ln w)}{-1}$$

$$= \ln w$$

$\longrightarrow$

Question 3

- CRRA $\longrightarrow$ DARA

From the class notes:

$$R_A \downarrow = \frac{R_R}{w} = \frac{\gamma}{w \uparrow}$$

- CRRA $\not\longleftarrow$ DARA

There are many utility functions that exhibit DARA but not CRRA.

For example, the quadratic utility function

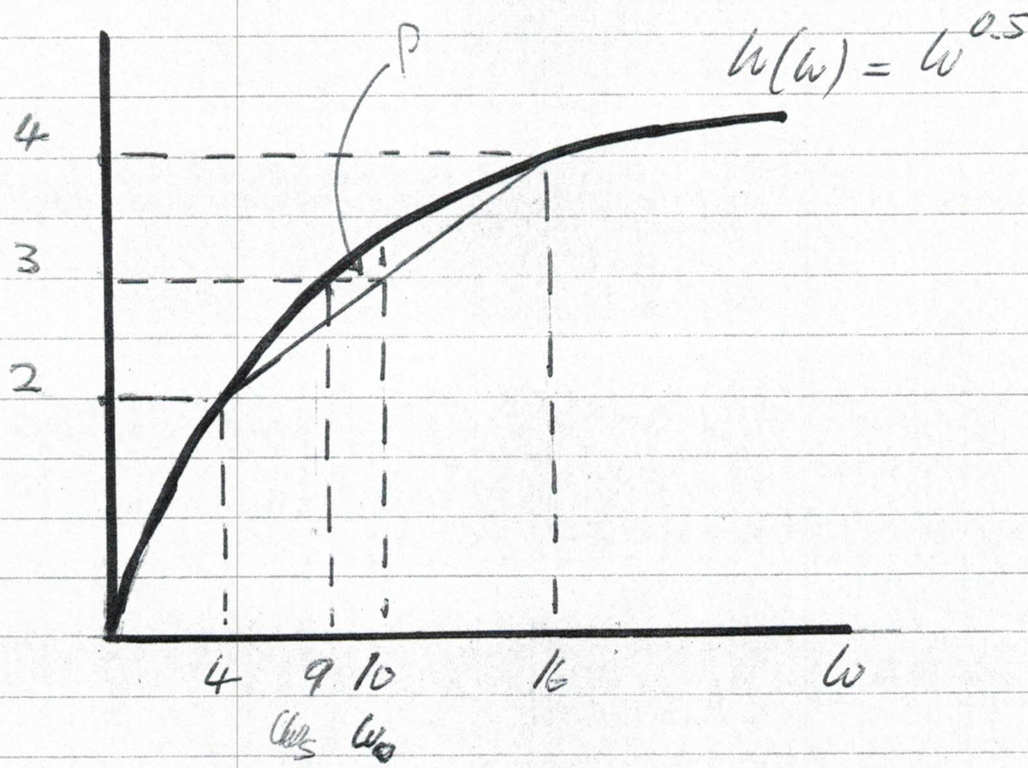
$$U = aw - bw^2 \qquad a, b > 0$$

exhibits DARA (when wealth is not too large) but not CRRA.

$$R_A \downarrow = -\frac{u''(w)}{u'(w)} = -\frac{-2b}{a - 2bw \uparrow} \qquad (a > 2bw)$$

$$R_R = -\frac{u''(w)w}{u'(w)} = \frac{2b}{a - 2bw}\, w$$

Question 4



a) $w_s^{0.5} = 3$

$w_s = 3^2 = 9$

$P = w_0 - w_s = 10 - 9 = 1$

b) $u' = 0.5\, w^{-0.5}$

$u'' = -0.25\, w^{-1.5}$

$$R_A = -\frac{u''}{u'} = -\frac{(-0.25\, w^{-1.5})}{0.5\, w^{-0.5}}$$

$$= \frac{\frac{1}{4}}{\frac{1}{2}} w^{-1} = \frac{1}{2w} = \frac{1}{20}$$

$E(x) = -6 \times 0.5 + 6 \times 0.5 = 0$

$$\sigma_x^2 = [-6-0]^2\, 0.5 + [6-0]^2\, 0.5$$

$$= 18 + 18 = 36$$

$$p \approx \frac{1}{2} \sigma_x^2 R_A = \frac{1}{2}\, 36\, \frac{1}{20}$$

$$= \frac{36}{40} = 0.9$$

c) $MWP = \underbrace{|E(x)|}_{\text{fair premium}} + p = 0 + 1 = 1$

Question 5

The consumer's utility function is:

$$v(w) = \ln w$$

$$v'(w) = \frac{1}{w} \left[= w^{-1} \right]$$

$$v''(w) = -w^{-2} = -\frac{1}{w^2}$$

The consumer's risk premium is:

$$\rho \approx -\frac{1}{2} \sigma_x^2 \underbrace{\frac{v''(w_0)}{v'(w_0)}}_{R_A} = -\frac{1}{2} \sigma_x^2 \frac{-\frac{1}{w_0^2}}{\frac{1}{w_0}}$$

$$= -\frac{1}{2} \sigma_x^2 \left(-\frac{1}{w_0} \right) = \frac{1}{2} 50{,}000^2 \frac{1}{500{,}000}$$

$$= \$ 2500$$

As the expected change in wealth is zero, $(EX = 0)$, the consumer would be willing to pay $\rho = \$2500$ to avoid the risk.

Question 6

The farm's wheat production X is a random variable with:

$$E(x) = 400 \times 1.8 = 720\ t$$
$$\sigma_x^2 = 400^2 \times 0.7^2 = 78{,}400\ t^2$$
$$\sigma_x = \sqrt{78{,}400} = 280\ t$$

The coefficient of variation is dimensionless:

$$\tilde{\sigma}_x = \frac{\sigma_x}{E(x)} = \frac{280}{720}\frac{t}{t} = 0.388889$$

The farmer's relative risk premium is

$$\rho^* \approx \frac{1}{2}\tilde{\sigma}_x^2 R_R$$
$$\approx \frac{1}{2} \times 0.388889^2 \times 1.5 = 0.1134$$

The farmer would be willing to give away 11.3% of income for multiperil crop insurance.

(Although hail insurance emerged in Europe in the 19th century, there still exists no private multi-peril crop insurance that covers hail, drought and other perils. In several countries governments provide multi-peril crop insurance that includes drought. The subsidy is substantial if public insurance is available at – say – 3 % and farmers would be willing to pay ~~7.7%~~ 11.3%.)

8-10

Question 7

- initial wealth (w_0)
- possible monetary loss (x)
- probability of loss event.

See next question.

8-11

Question 8

a) Without insurance, expected utility is:

$$EU = 0.998 \ln(100{,}000) + 0.002 \ln(100{,}000 - 20{,}000)$$
$$= 11.51248$$

The certainty equivalent is:

$$\ln W_S = 11.51248$$

$$W_S = e^{11.51248} = \underline{\underline{\$\,99{,}955.38}}$$

b) The expected loss is:

$$EX = x\pi = 20{,}000 \times 0.002 = \$\,40$$

Then, the consumer's risk premium is:

$$\rho = \underbrace{EW}_{W_0 - EX} - \underbrace{U^{-1}(EU)}_{W_S}$$

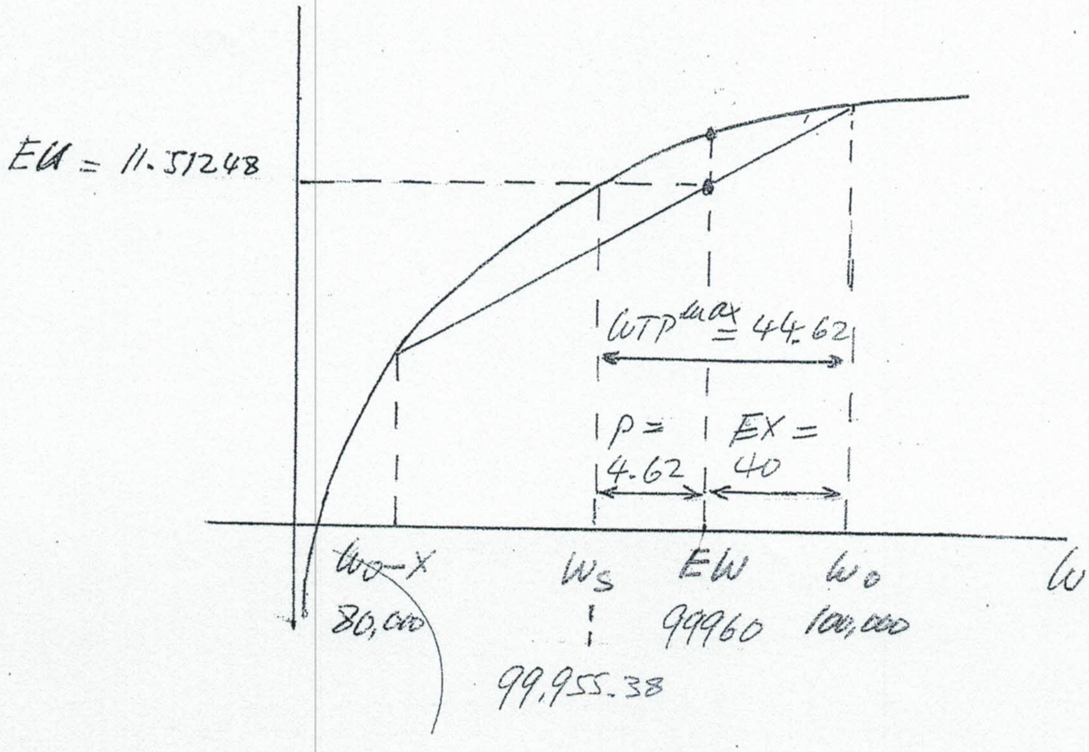
$$= W_0 - EX - W_S$$

$$= 100'000 - 40 - 99{,}955.38 = \underline{\underline{\$\,4.62}}$$

The maximum willingness to pay for insurance is:

$$WTP^{max} = 40 + 4.62 = \underline{\underline{\$\,44.62}}$$

8 - 12



Summary

Definition of risk premium ρ:

$$EU = U(EW - \rho)$$

$$0.998 \ln 100{,}000 + 0.002 \ln(100{,}000 - 20{,}000) = \ln(100{,}000 - 40 - \rho)$$

$$11.51248 = \ln(99960 - \rho)$$

$$e^{11.51248} = 99960 - \rho$$

$$\rho = \underline{\underline{\$\,4.62}}$$

8-13

c) The curvature of the logarithmic utility function is a bit low. The coefficient of relative risk aversion equals only 1.

An only moderately risk-averse person does not pay much for insurance.